Transport in Porous Media **18:** 15–36, 1995.

An Integral Equation Formulation for the Unconfined Flow of Groundwater with Variable Inlet Conditions*

Z.-X. CHEN[1], G. S. BODVARSSON[1], P. A. WITHERSPOON[1], and Y. C. YORTSOS[2] **
[1]*Lawrence Berkeley Laboratory, University of California, Berkeley, CA 94720, U.S.A.*
[2]*Department of Chemical Engineering, University of Southern California, Los Angeles, CA 90089-1211, U.S.A.*

(Received: 13 December 1993; in final form: 19 July 1994)

Abstract. We combine an integral equation formulation with a hodograph transformation to solve self-similar problems describing the unconfined flow of groundwater with variable inlet conditions. A class of new semi-analytical solutions is obtained for both rectilinear and radial flow geometries. The solutions are in general agreement with those derived by Barenblatt, although there are some discrepancies for the case of radial flow. The formulation presented provides additional analytical insight, and for computational purposes is simpler than Barenblatt's. In addition, the method proposed can be successfully used for the solution of a host of other nonlinear problems that admit self-similarity.

Key words: Analytical solution, unconfined flow, groundwater.

Nomenclature

a	parameter defined in (3) $[L/T]$
f	relative total discharge function for the case of constant head
g	gravitational constant $[L/T^2]$
h	hydraulic head $[L]$
H	dimensionless hydraulic head
k	permeability $[L^2]$
K	hydraulic conductivity $[L/T]$
Q	total discharge per unit width $[L^2/T]$
R	relative total discharge function
r	radial distance $[L]$
t	time $[T]$

* This paper, first prepared in 1991, was left incomplete due to the untimely passing of its first author, Z.-X. Chen. It is dedicated to his memory.

** Author to whom correspondence should be addressed.

v dimensionless total discharge
V cumulative volume $[L^3]$
w total discharge per unit head
x distance $[L]$
α exponent of time variation of inlet head
λ dimensionless constant defined in (9)
μ viscosity $[M/TL]$
ξ similarity variable
ρ density $[M/L^3]$
ϕ porosity

Introduction

Displacement processes in porous media are inherently unsteady, multi-dimensional and nonlinear. Although the fundamental insight at the pore-level requires the use of statistical physics methods (such as percolation and DLA, e.g., see Lenormand, 1990), the macroscopic description is often adequate with the use of partial differential equations (particularly in homogeneous media and stable displacements). Under certain conditions, additional simplifications are possible that render these problems tractable by relatively simple computational techniques. The reduction in dimensionality, obtained when the reservoir is long and narrow, is such a simplification. More precisely, this is valid when the ratio $R_L = \frac{L}{H}\sqrt{\frac{k_V}{k_H}}$ is large, where L and H denote the length and thickness of the reservoir, respectively, and k_V and k_H are horizontal and vertical permeabilities (e.g. Lake, 1989 and Yortsos, 1991).

Such an approach has been taken in both the groundwater and the reservoir engineering literatures. In the field of groundwater flow, the approximation is known as the *Dupuit assumption.* Boussinesq (1904) was the first to use it in the description of unconfined flow. The resulting equation is a form of a nonlinear diffusion equation, with a diffusion coefficient that vanishes for a particular value of the dependent variable. Similar equations were found to describe the isothermal, unsteady-state flow of an ideal gas through porous media (Liebenzon, 1929). In the reservoir engineering literature, the approximation is commonly known as the *Vertical Equilibrium (VE)* approximation (the term vertical actually denoting transverse to flow) (Lake, 1989). Yortsos (1991) has recently classified the various regimes that emerge from the application of VE in displacement processes. Nonlinear diffusion equations also arise with the use of the traditional relative permeability and capillary pressure formalisms to describe 1-D displacement problems.

In the case of homogeneous media, nonlinear diffusion equations typically admit a similarity transformation. This further reduces the dimensionality of the problem because it allows time, space and dependent variables to be combined in

unsteady, multi-dimensional
: pore-level requires the use
DLA, e.g., see Lenormand,
with the use of partial dif-
a and stable displacements).
e possible that render these
l techniques. The reduction
; and narrow, is such a sim-
:io $R_L = \frac{L}{H}\sqrt{\frac{k_V}{k_H}}$ is large,
reservoir, respectively, and
.g. Lake, 1989 and Yortsos,

vater and the reservoir engi-
: approximation is known as
to use it in the description of
onlinear diffusion equation,
ılar value of the dependent
: isothermal, unsteady-state
on, 1929). In the reservoir
known as the *Vertical Equi-*
denoting transverse to flow)
various regimes that emerge
. Nonlinear diffusion equa-
permeability and capillary
blems.
ffusion equations typically
s the dimensionality of the
variables to be combined in

groups and to transform the partial differential equation into an ordinary differential equation. Certain, but not all, initial and boundary conditions allow for a similarity transformation to be applied. Typically, the result of a similarity transformation is a two-point boundary value problem in a semi-infinite interval $[0, \infty]$, the solution of which requires in general the application of a shooting method. Some self-similar problems for the unconfined flow of groundwater were formulated and solved in the classical papers of Polubarinova–Kochina (1948, 1949) and Barenblatt (1952a–b, 1954). Many of these results are also summarized in the recent book by Barenblatt *et al.* (1990). Unfortunately, except for some very special cases, the closed-form solution of the resulting boundary value problem is not possible. Nonetheless, the similarity reduction is very useful, because it allows for many interesting results to be obtained, for example regarding the structure at the front, its speed of propagation and some other nonlinear characteristics. Certain of these properties are distinctly pertinent to nonlinear flows, and cannot be simply inferred from linearization (Barenblatt, 1952a, 1952b).

Sometimes, a useful alternative to the shooting method is provided by recasting the problem into an integral equation. This technique has been frequently used in the area of nonlinear diffusion. Applications to porous media problems were recently described by McWhorter (1990), McWhorter and Sunada (1990), and Chen *et al.* (1990, 1991a, 1991b), who suggested the reduction of the problem into an integral equation based on some physical considerations. While straightforward, the physical approach is somewhat restrictive. A careful analysis of the problem shows that the reduction to an integral equation is possible for more general conditions, by introducing a *hodograph transformation.* This method, which interchanges dependent and independent variables, is not subject to the possible limitations of the more intuitive approaches and provides an almost algorithmic scheme for the solution of a variety of nonlinear problems. This paper presents an application of this approach to the solution of the unconfined flow of groundwater in both rectilinear and radial geometries. With an obvious change in notation, the approach applies equally well to the isothermal flow of gas in porous media, as well as to a variety of other nonlinear problems.

Rectilinear Flow

Consider a rectilinear, semi-infinite and horizontal porous medium, which rests on an impermeable base and it is connected to a canal with a vertical boundary (Figure 1). Initially, the hydraulic head of the groundwater in the porous medium is assumed to be equal to zero, the pore space being occupied only by gas (air) of negligible viscosity. At the boundary ($x = 0$), the hydraulic head in the canal varies with time according to the power law

$$h(0,\ t) = \sigma t^{\alpha} \tag{1}$$

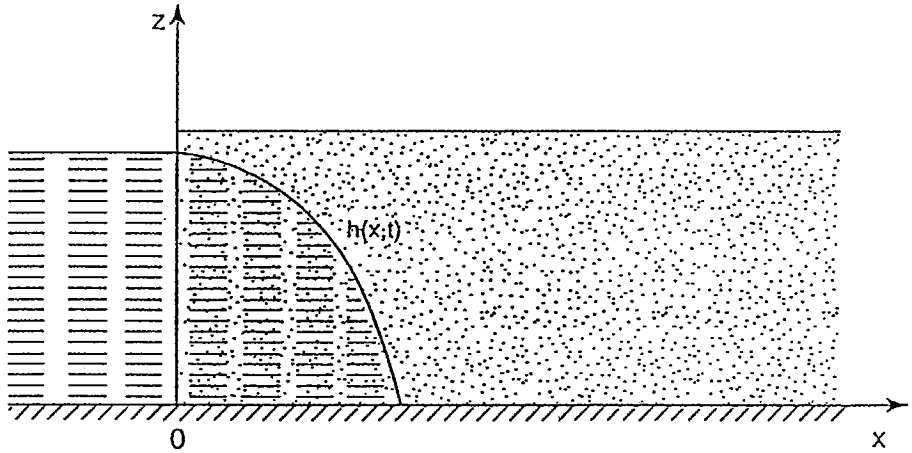


Fig. 1. Schematic of unconfined flow of groundwater.

where σ is a scaling parameter and α is a constant in the range $[-1/3, \infty]$ (see below for the reason for this restriction). Use of the Dupuit assumption (see Bear, 1972, Yortsos, 1991) allows for an one-dimensional description, where the hydraulic head, $h(x, t)$, satisfies the Boussinesq equation

$$\frac{\partial h}{\partial t} = a\frac{\partial^2 h^2}{\partial x^2} \tag{2}$$

and we have denoted

$$a = \frac{k\rho g}{2\phi\mu}. \tag{3}$$

Nonlinear, parabolic equations of this type arise in many applications (e.g. Aronson, 1985, van Duijn and Peletier, 1992) and they belong to the general class referred to as *the porous media equation*. In the present problem, we are interested in solving (2) subject to the boundary condition (1) and the following initial and boundary conditions

$$h(x, 0) = 0 \tag{4}$$

$$h(\infty, t) = 0. \tag{5}$$

Because of the power-law form of (1) a similarity transformation can be applied. We introduce the dimensionless groups (Barenblatt, 1952a)

$$\xi = x\sqrt{\frac{\alpha + 1}{a\sigma t^{\alpha+1}}} \tag{6}$$

and

$$H(\xi;\, \lambda) = \frac{h}{\sigma t^{\alpha}} \tag{7}$$

to transform (2) to the second-order ordinary differential equation

$$\frac{\mathrm{d}^2 H^2}{\mathrm{d}\xi^2} + \frac{1}{2}\xi\frac{\mathrm{d}H}{\mathrm{d}\xi} - \lambda H = 0, \tag{8}$$

where we defined

$$\lambda \equiv \frac{\alpha}{1+\alpha} \quad (-1/2 < \lambda < 1). \tag{9}$$

Parameter λ is another measure of the variation of the hydraulic head at the boundary. Equation (8) is to be solved subject to the conditions

$$H(0;\, \lambda) = 1 \tag{10}$$

$$H(\infty;\, \lambda) = 0. \tag{11}$$

The self-similar formulation (8)–(11) is originally due to Barenblatt (1952a–b, 1954). He proceeded with a thorough analytical study of the integral curves of (8) and has suggested an effective method for the calculation of H. Barenblatt revealed many of the basic characteristics of this solution, particularly the finite value of the velocity of the propagating front.

Recently, Chen *et al.* (1991b) were able to provide an alternative method for the solution of (8) in the special case $\lambda = 0$, based on an integral equation formulation. Their approach was guided from physical considerations. Use of a careful analysis, however, reveals that with the application of a hodograph transformation, the reduction to an integral equation is possible for arbitrary λ. Hodograph transformations have been successfully used in a variety of nonlinear problems in homogeneous media (see, for example, Fokas and Yortsos, 1982, Shankar and Yortsos, 1983 and Clarkson *et al.*, 1989). This technique will be subsequently applied to this problem.

INTEGRAL EQUATION FORMULATION

By interchanging dependent and independent variables and by denoting $\xi \equiv \xi(H)$, problem (8)–(11) now reads

$$\frac{\mathrm{d}}{\mathrm{d}H}\left[\frac{H}{\frac{\mathrm{d}\xi}{\mathrm{d}H}}\right] + \frac{1}{4}\xi - \frac{1}{2}\lambda H\frac{\mathrm{d}\xi}{\mathrm{d}H} = 0 \tag{12}$$

with

$$\xi(1) = 0 \tag{13}$$

and

$$\xi(0) = \infty \quad \left(\text{and } \frac{\mathrm{d}\xi}{\mathrm{d}H} \to \infty,\ H \to 0\right). \tag{14}$$

The essential condition for the validity of this transformation is that $\mathrm{d}\xi/\mathrm{d}H$ does not change sign in the interval of interest, so that the map $\xi \leftrightarrow H$ remains one-to-one. For the particular problem under consideration, the constraint is also physically meaningful. Time variations different than (1), however, generally violate this condition. To proceed with the solution, we next introduce the auxiliary variable ω, defined by

$$\frac{\mathrm{d}\omega}{\mathrm{d}H} = \frac{1}{4}\xi - \frac{1}{2}\lambda H \frac{\mathrm{d}\xi}{\mathrm{d}H} \tag{15}$$

the physical meaning of which will be discussed shortly. In terms of ω we have

$$\frac{\mathrm{d}}{\mathrm{d}H}\left[\frac{H}{\frac{\mathrm{d}\xi}{\mathrm{d}H}}\right] + \frac{\mathrm{d}\omega}{\mathrm{d}H} = 0, \tag{16}$$

which can be directly integrated. We get

$$\frac{\mathrm{d}\xi}{\mathrm{d}H} = \frac{H}{C_1 - \omega}, \tag{17}$$

where the constant C_1 follows from boundary condition (14)

$$C_1 = \omega(0). \tag{18}$$

Equivalently, and without loss of generality, we may take $\omega(0) = 0$, as shown below. Substitution in (17), followed with a further integration, and use of (13) results in

$$\xi = \int_1^H \frac{\beta}{\omega(0) - \omega(\beta)}\,\mathrm{d}\beta. \tag{19}$$

This expression for ξ can be substituted in (15) to yield a differential equation for ω

$$\frac{\mathrm{d}\omega}{\mathrm{d}H} = \frac{1}{4}\int_1^H \frac{\beta}{\omega(0) - \omega(\beta)}\,\mathrm{d}\beta - \frac{1}{2}\lambda \frac{H^2}{\omega(0) - \omega}. \tag{20}$$

The latter can be directly integrated. In fact, the resulting double integral can be conveniently rearranged to lead to the desired integral equation

$$\omega = \frac{1}{4}\int_1^H \frac{(H-\beta)\beta}{\omega(0)-\omega(\beta)}\,\mathrm{d}\beta - \frac{1}{2}\lambda\int_1^H \frac{\beta^2}{\omega(0)-\omega(\beta)}\,\mathrm{d}\beta + C_2, \tag{21}$$

where

$$C_2 = \omega(0) + \frac{1}{4}(1+2\lambda)\int_1^0 \frac{\beta^2}{\omega(0)-\omega(\beta)}\,\mathrm{d}\beta. \tag{22}$$

Equivalently, we may note that $\omega(0)$ is arbitrary, since it can be removed by defining the new variable

$$v(H;\,\lambda) \equiv \omega - \omega(0), \tag{23}$$

which, when substituted in (21) and (22), yields an equivalent integral equation for v

$$v = \frac{1}{4}\left[(1+2\lambda)\int_H^1 \frac{\beta^2}{v(\beta)}\,\mathrm{d}\beta + H\int_H^1 \frac{\beta}{v(\beta)}\,\mathrm{d}\beta\right]. \tag{24}$$

In terms of v, equation (19) reads

$$\xi = \int_H^1 \frac{\beta}{v(\beta)}\,\mathrm{d}\beta. \tag{25}$$

Thus, for any value of λ, we have reduced the problem into one of solving the integral equation (24). Equations (24) and (25) implicitly provide the solution to the problem.

Before we proceed with the solution of (24), we shall comment on the behavior near $H = 0$. As can be easily shown by direct substitution, an expansion of v near $H = 0$ shows the leading-order behavior

$$v \sim H + O(H^2). \tag{26}$$

This implies that all integrals in (24) and (25) are convergent, and that ξ takes a finite value ξ_0 at $H = 0$. Equivalently stated, the hydraulic head becomes zero at a finite location. This distinct property of nonlinear flow (which, for example, is opposite to the infinite propagation speed of linear diffusion fronts) was discovered by Barenblatt (1952a, 1952b) and theoretically proven by Barenblatt and Vishek (1956) (see also recent developments in Aronson, 1985). Because of this property, the self-similar solutions developed are also valid for a *finite* medium, as long as the front has not reached the outer boundary.

To solve the integral equation (24) we follow an iterative procedure using a test function as an initial guess. Physical considerations can guide us to the selection of a good initial estimate. To this order, we shall relate v to the physical quantity of the relative total discharge function.

THE RELATIVE TOTAL DISCHARGE FUNCTION

Following Darcy's law and the Dupuit assumption, the total discharge per unit width, Q, can be expressed as

$$Q = -Kh\frac{\partial h}{\partial x}, \tag{27}$$

where $K = k\rho g/\mu$ is the hydraulic conductivity. In terms of H and ξ, we obtain (see also Barenblatt *et al.* (1990) but also note some typographical errors)

$$Q = -\frac{K}{2}\left[\frac{\sigma^3}{a}(\alpha+1)t^{3\alpha-1}\right]^{1/2}\frac{\mathrm{d}H^2}{\mathrm{d}\xi}. \tag{28}$$

Combining (17), (18) and (23), we have

$$v = -\frac{1}{2}\frac{\mathrm{d}H^2}{\mathrm{d}\xi}, \tag{29}$$

thus, we can relate v to Q by

$$v = \frac{Q}{K\left[\frac{\sigma^3}{a}(\alpha+1)t^{3\alpha-1}\right]^{1/2}}. \tag{30}$$

It is evident from the above expression, that v represents a dimensionless *total discharge*. Proceeding further, we can define the *relative total discharge* function, $R(H;\,\lambda)$, by normalizing with the total discharge at the inlet, to obtain

$$R(H;\,\lambda) \equiv \frac{v(H;\,\lambda)}{v(1;\,\lambda)} \quad (\lambda \neq -1/2) \tag{31}$$

or

$$R(H;\,\lambda) = \frac{4v(H;\,\lambda)}{(1+2\lambda)\int_0^1 \frac{\beta^2}{v(\beta)}\,\mathrm{d}\beta} \quad (\lambda \neq -1/2). \tag{32}$$

In the special case $\lambda = 0$, which corresponds to a constant hydraulic head at the inlet ($\alpha = 0$), Chen *et al.* (1991b) derived an integral equation involving the relative

erative procedure using a test
can guide us to the selection
ate v to the physical quantity

the total discharge per unit

(27)

terms of H and ξ, we obtain
typographical errors)

(28)

(29)

(30)

resents a dimensionless *total*
tive total discharge function,
the inlet, to obtain

(31)

). (32)

tant hydraulic head at the inlet
quation involving the relative

total discharge function, which in their paper they denoted by f, $f(H) \equiv R(H;0)$. These authors found that f satisfies the equation

$$f(H) = 1 + \frac{\int_H^1 \frac{(H-\beta)\beta}{f(\beta)}\,\mathrm{d}\beta}{\int_0^1 \frac{\beta^2}{f(\beta)}\,\mathrm{d}\beta} \tag{33}$$

and that

$$\xi = \frac{2}{Q_D(0)} \int_H^1 \frac{\beta}{f(\beta)}\,\mathrm{d}\beta, \tag{34}$$

where they used the dimensionless total discharge constant

$$Q_D(0) = \left[\int_0^1 \frac{\beta}{f(\beta)}\,\mathrm{d}\beta\right]^{1/2}. \tag{35}$$

It can be readily shown that their results are contained in the present formulation in the limit $\lambda = 0$. Indeed, by using (32), we find the expression

$$f(H) = \frac{4v(H;0)}{\int_0^1 \frac{\beta^2}{v(\beta)}\,\mathrm{d}\beta}, \tag{36}$$

which if substituted in (33)–(35), yields (24) and (25), as expected.

We can proceed with the use of the more physical variable R instead of v. For example, we can rearrange (31) to read as

$$v(H;\,\lambda) = \tfrac{1}{2}(1+2\lambda)^{1/2}\left[\int_0^1 \frac{\beta^2}{R(\beta)}\,\mathrm{d}\beta\right]^{1/2} R \tag{37}$$

and obtain, after some algebra, the equivalent integral equation

$$R = \frac{(1+2\lambda)\int_0^H \frac{\beta^2}{R(\beta)}\,\mathrm{d}\beta + H\int_H^1 \frac{\beta}{R(\beta)}\,\mathrm{d}\beta}{(1+2\lambda)\int_0^1 \frac{\beta^2}{R(\beta)}\,\mathrm{d}\beta} \tag{38}$$

now expressed in terms of the physical quantity R. The equation for ξ follows directly

$$\xi = \frac{2}{\left[(1+2\lambda)\int_0^1 \frac{\beta^2}{R(\beta)}\,\mathrm{d}\beta\right]^{1/2}} \int_H^1 \frac{\beta}{R(\beta)}\,\mathrm{d}\beta. \tag{39}$$

Thus, by following the hodograph approach, we have arrived at an integral equation in terms of a physical variable. To solve (38) we note that, as expected, $R(0;\,\lambda) = 0$

and $R(1;\ \lambda) = 1$. Thus, a reasonable initial guess for R would be $R_0(H;\ \lambda) = H$. As will be shown below, the solution of the integral equation (38) converges rapidly with this initial guess. Numerical results are presented in the next section.

SPECIAL CASE $\lambda = -1/2$

The special case $\lambda = -1/2$ is worth examining because it admits the exact solution derived by Barenblatt *et al.* (1972)

$$H(\xi;\ -1/2) = \begin{cases} 1 - \xi^2/8 & (0 \leqslant \xi \leqslant \sqrt{8}) \\ 0 & (\xi \geqslant \sqrt{8}). \end{cases} \tag{40}$$

For this value we have $v(1;\ -1/2) = 0$. Because the total discharge at the inlet vanishes, however, R becomes unbounded. In this case we must revert to equations (24) and (25) for v, instead of attempting to solve for R (equations (38) and (39)). To derive Barenblatt's result with the present formulation, we note that for $\lambda = -1/2$, Equation (24) yields

$$\frac{v}{H} = \frac{1}{4}\int_H^1 \frac{\beta}{v(\beta)}\,\mathrm{d}\beta, \tag{41}$$

which, using the change of variables,

$$w = \frac{v}{H} \tag{42}$$

can be converted to the simpler equation

$$w = \frac{1}{4}\int_H^1 \frac{\mathrm{d}\beta}{w(\beta)}. \tag{43}$$

By inspection, a solution to the latter is

$$w = \frac{1}{\sqrt{2}}(1 - H)^{1/2} \tag{44}$$

thus,

$$v = \frac{1}{\sqrt{2}}H(1 - H)^{1/2} \tag{45}$$

and, from equation (25),

$$\xi = 2\sqrt{2}(1 - H)^{1/2}. \tag{46}$$

This is identical to Barenblatt's result (40).

R would be $R_0(H;\, \lambda) = H$.
uation (38) converges rapidly
d in the next section.

se it admits the exact solution

(40)

ne total discharge at the inlet
e we must revert to equations
(equations (38) and (39)). To
n, we note that for $\lambda = -1/2$,

(41)

(42)

(43)

(44)

(45)

(46)

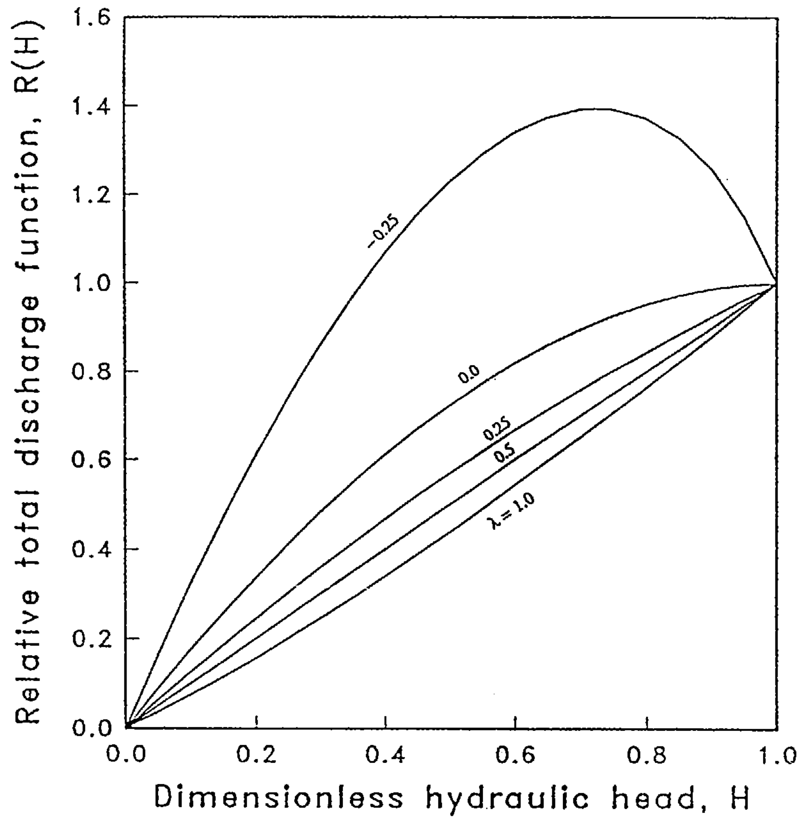


Fig. 2. The relative total discharge function $R(H)$ for various values of λ.

NUMERICAL RESULTS

Numerical calculations were carried out for various values of the parameter λ in the range $-1/2$ to 1. With the use of the initial guess, R_0, the iterative process for solving (38) was found to converge quite rapidly. The results are in exact agreement with those reported by Barenblatt (1954) and are illustrated in Figure 2 through 6. In these Figures, ξ_0 is the dimensionless front location. The gradient at the inlet, which is related to $v(1)$ via (29), can also be related to the cumulative volume, $V(t)$

$$V(t) = \int_0^{\infty} \phi h(x,\, t)\, \mathrm{d}x = -\phi \left[\frac{a\sigma^3}{\alpha+1} t^{3\alpha+1} \right]^{1/2} \int_0^{\infty} H(\xi;\, \lambda)\, \mathrm{d}\xi. \tag{47}$$

Indeed, by integrating (8) we obtain

$$\int_0^{\infty} H(\xi;\, \lambda)\, \mathrm{d}\xi = -\frac{2}{1+2\lambda} \left. \frac{\mathrm{d}H^2}{\mathrm{d}\xi} \right\|_{\xi=0} \tag{48}$$

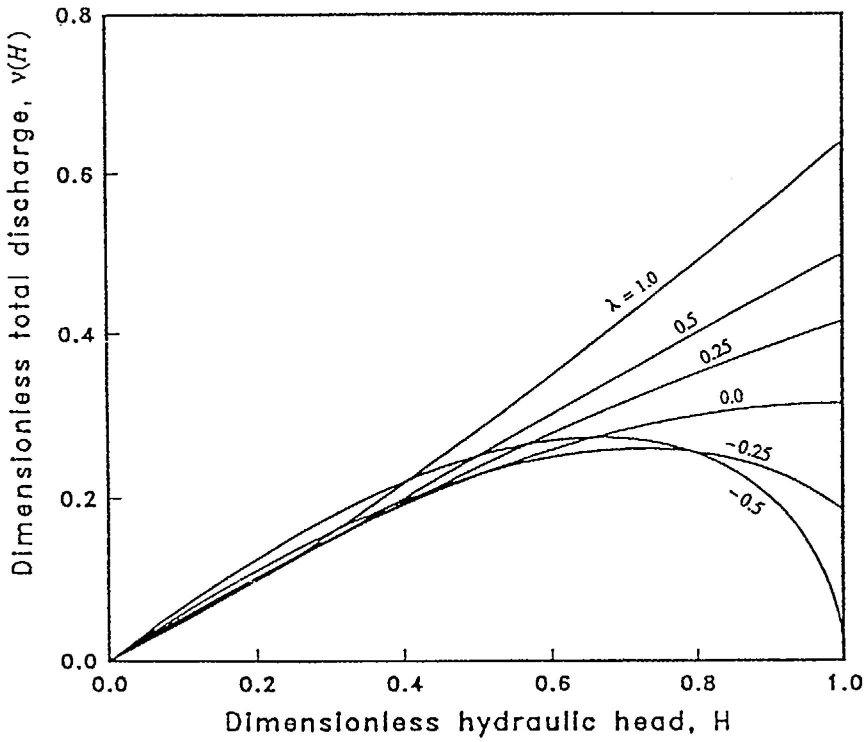


Fig. 3. The function $v(H)$ for various values of λ.

or, in terms of v,

$$\int_0^\infty H(\xi;\ \lambda)\,\mathrm{d}\xi = \int_0^1 \frac{\beta^2}{v(\beta)}\,\mathrm{d}\beta = \frac{4}{1+2\lambda}v(1;\ \lambda). \tag{49}$$

Thus, the quantity $V_D \equiv \int_0^\infty H(\xi;\ \lambda)\,\mathrm{d}\xi$ can be regarded as the dimensionless cumulative volume in the medium.

Curves of the relative discharge function for various values of λ are illustrated in Figure 2. All curves emanate from the origin and end at (1, 1). For $\lambda > 0$, they monotonically increase with increasing H. At $\lambda = 1/2$, the solution coincides with the diagonal. This can be verified by comparing with the exact results (Barenblatt *et al.*, 1972)

$$H(\xi;\ 1/2) = \begin{cases} 1 - \xi/2 & (0 \leqslant \xi \leqslant 2) \\ 0 & (\xi \geqslant 2), \end{cases} \tag{50}$$

which imply a linear relationship between R and H. The curves for $0 < \lambda < 1/2$ are convex, those for $1/2 < \lambda < 1$ are concave. For $\lambda > 0$, the maximum total discharge occurs at the inlet. However, for $\lambda < 0$, the relative total discharge

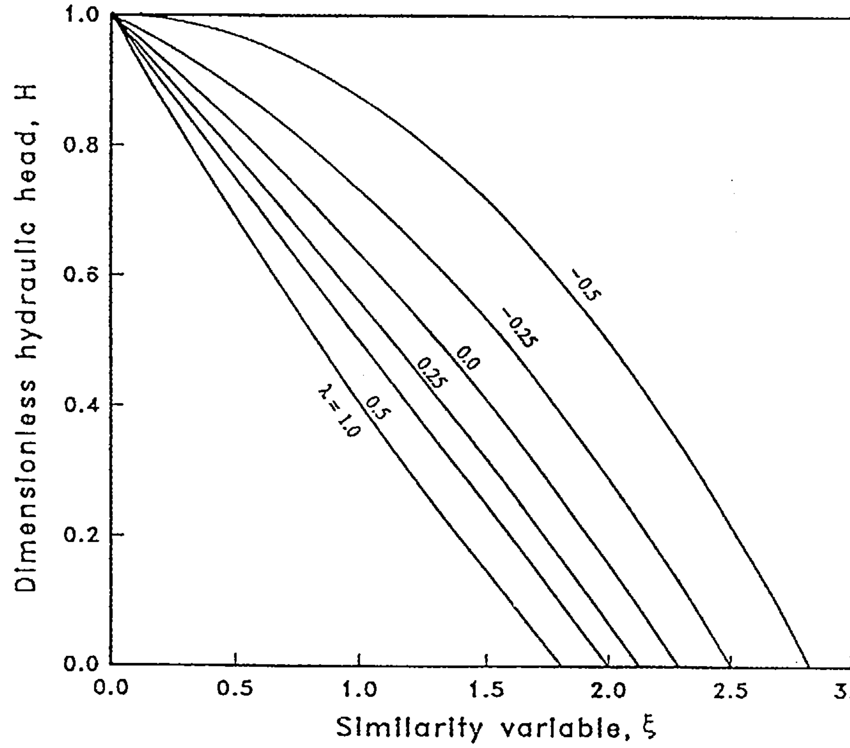


Fig. 4. The distribution of the dimensionless hydraulic head H for various values of λ.

function is not monotonic. The curves become convex with a maximum occurring somewhere in the interval (0, 1) and they expand upwards as λ decreases. When $\lambda = -1/2$, we find $R = \infty$ everywhere in the interval (0, 1) of the H axis. Physically, this implies that, following instantaneous infiltration at $t = 0$, there is no further water flow into the porous medium (see also Figure 5).

Figure 3 shows plots of the function $v(H;\ \lambda)$. As predicted, the slopes at the origin are finite. For $-1/2 < \lambda$, the function is positive and does not change sign, suggesting that the hodograph transformation is well posed. However, for $\lambda = -1/2$, v vanishes at $H = 1$ (in fact, its derivative is unbounded there), signalling ill-posedness. Indeed, for smaller values, $\lambda < -1/2$, v changes sign in the interval (0, 1), thus violating the monotonicity requirement of the transformation and invalidating the approach taken in that region. Figure 4 shows dimensionless hydraulic head profiles for various values of λ. All curves terminate at the finite location ξ_0, ahead of which the porous medium is at its initial state. The dependence of ξ_0 on λ is rather weak as shown in Figure 5. On the other hand, the dependence of $(-\frac{\mathrm{d}H^2}{\mathrm{d}\xi}||_{\xi=0} \equiv 2v)$ on λ is stronger (Figure 5). As previously pointed out, for $\lambda = -1/2\ (\alpha = -1/3)$, the total discharge at the inlet vanishes for $t > 0$, and the volume of groundwater within the porous medium, which was infiltrated instan-

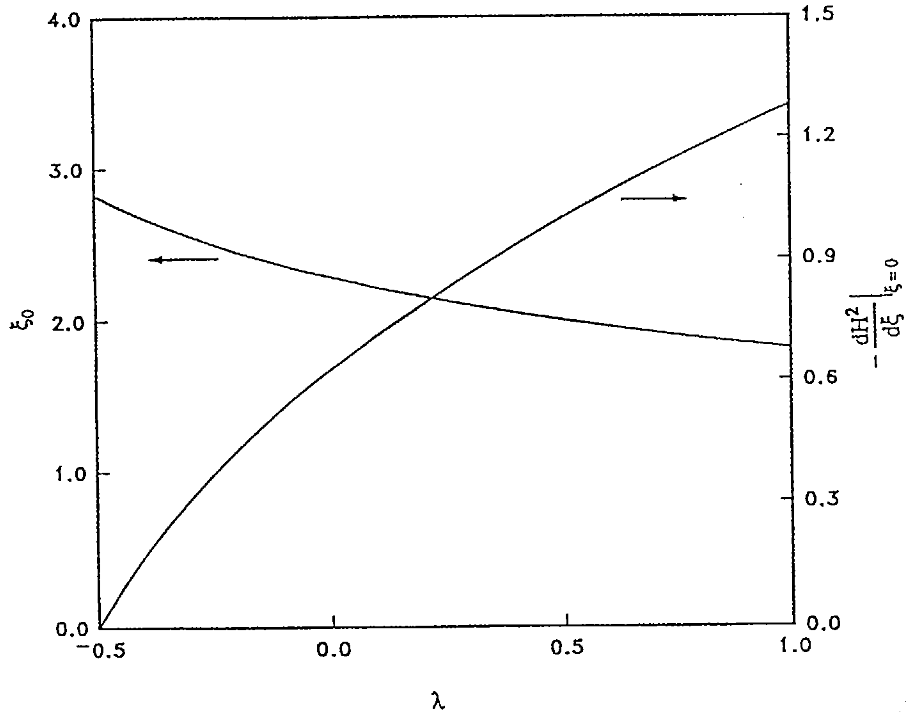


Fig. 5. The dependence of ξ_0 and the gradient of H^2 (which equals $2v$) on λ.

taneously at $t = 0$, remains constant throughout. The dimensionless cumulative volume, V_D, is plotted in Figure 6 in terms of the parameter λ. It is shown to decrease monotonically with increasing λ.

Radial Flow

A similar procedure can also be applied for the solution of the problem in radial geometry. Assuming axisymmetric unconfined Dupuit flow, the hydraulic head h satisfies

$$\frac{\partial h}{\partial t} = a\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial h^2}{\partial r}\right], \tag{51}$$

where r is the radial distance from the point source. For a similarity solution, we consider the problem posed by Barenblatt (1952a–b, 1954). Starting at $t = 0$, water is injected into an initially dry porous medium, at a variable rate obeying the power law:

$$Q(t) = \tau t^{\beta}, \tag{52}$$

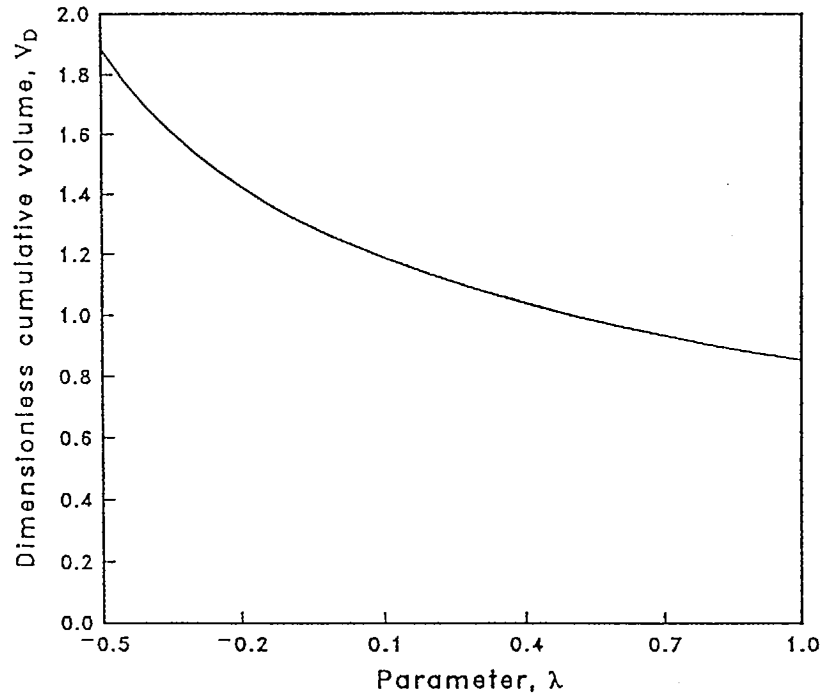


Fig. 6. The dependence of the dimensionless cumulative volume V_D on λ.

where $\tau > 0$ and $\beta > -1$. Again, constant injection rate corresponds to $\beta = 0$. Appropriate initial and boundary conditions for this problem are

$$h(r,\, 0) = 0 \tag{53}$$

$$h(\infty,\, t) = 0 \tag{54}$$

while (52) can be expressed as

$$\left[r \frac{\partial h^2}{\partial r} \right]_{r=0} = -\frac{\tau}{\pi K} t^{\beta}. \tag{55}$$

To proceed, we introduce the dimensionless hydraulic head H

$$H = \frac{h}{\sqrt{\frac{\tau}{\pi K} t^{\beta}}} \tag{56}$$

and the similarity variable

$$\xi = \left[\frac{\pi K (\beta + 2)^2}{4a^2 \tau t^{\beta+2}} \right]^{1/4} r, \tag{57}$$

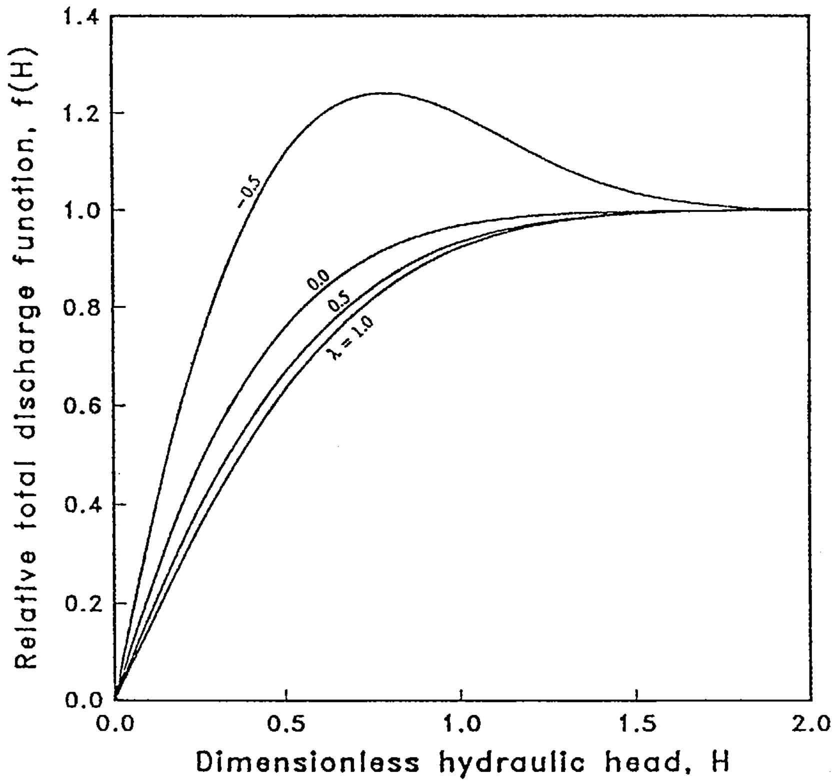


Fig. 7. The relative total discharge function $f(H)$ for various values of λ.

which reduce the above to the boundary value problem (Barenblatt, 1952a)

$$\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi\frac{\mathrm{d}H^2}{\mathrm{d}\xi}\right]+\frac{1}{2}\xi\frac{\mathrm{d}H}{\mathrm{d}\xi}-\lambda H=0 \tag{58}$$

with conditions

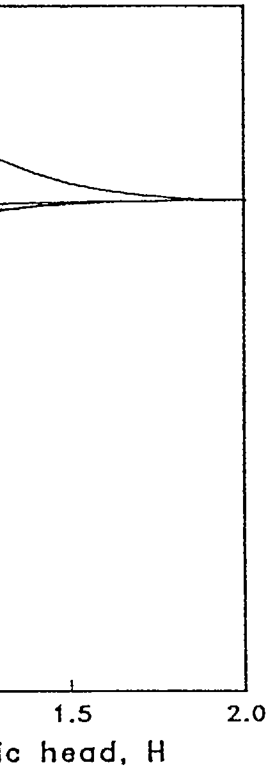
$$\xi\frac{\mathrm{d}H^2}{\mathrm{d}\xi}\bigg\|_{\xi=0}=-1 \tag{59}$$

and

$$H(\infty)=0, \tag{60}$$

where we defined

$$\lambda=\frac{\beta}{\beta+2}. \tag{61}$$



rious values of λ.

em (Barenblatt, 1952a)

(58)

(59)

(60)

(61)

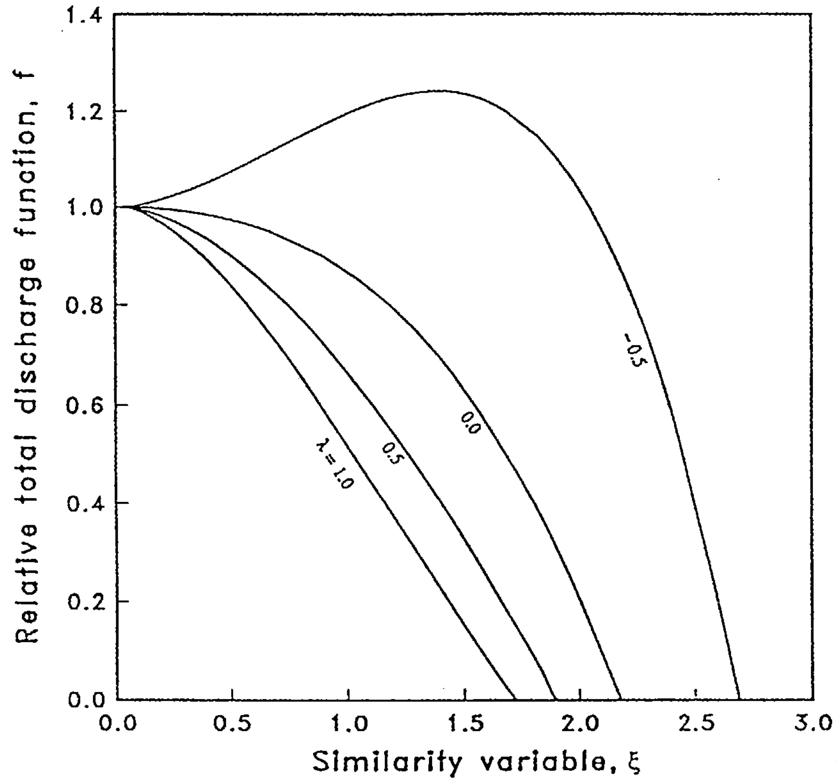


Fig. 8. The distribution of the relative total discharge function f for various values of λ.

As in the previous case, to solve (58) we will apply a hodograph transformation. Equations (58)–(60) are converted to

$$\frac{\mathrm{d}}{\mathrm{d}H}\left[\frac{2\xi H}{\frac{\mathrm{d}\xi}{\mathrm{d}H}}\right] + \frac{1}{2}\xi^2 - \lambda\xi H\frac{\mathrm{d}\xi}{\mathrm{d}H} = 0 \tag{62}$$

$$2\xi\left[\frac{H}{\frac{\mathrm{d}\xi}{\mathrm{d}H}}\right]\Bigg\|_{H=\infty} = -1 \tag{63}$$

and

$$\xi(0) = \infty. \tag{64}$$

We, next, define the auxiliary variable ω by taking

$$\frac{\mathrm{d}\omega}{\mathrm{d}H} = \frac{1}{2}\xi^2 - \lambda\xi H\frac{\mathrm{d}\xi}{\mathrm{d}H} \tag{65}$$

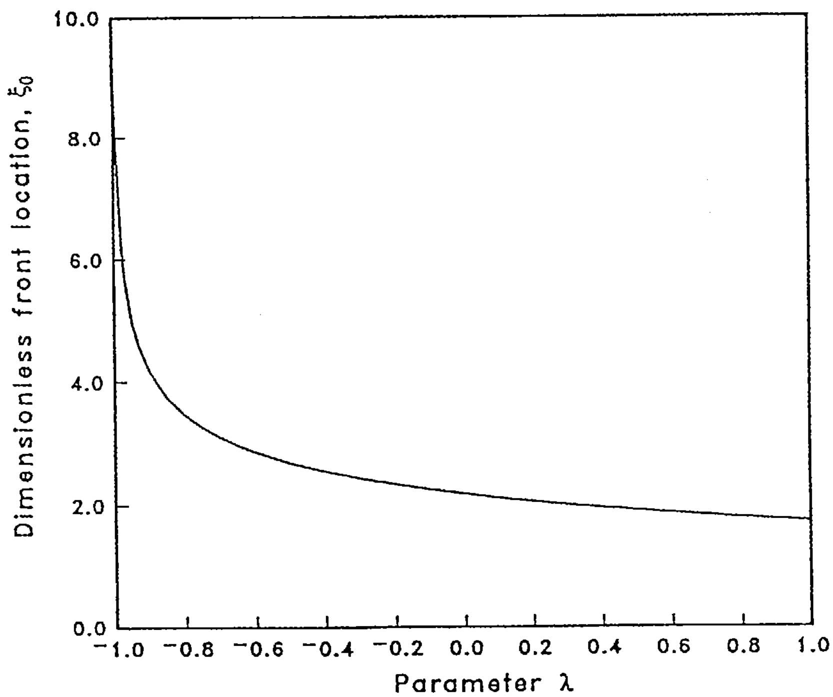


Fig. 9. The dimensionless front location ξ_0 as a function of λ.

and substitute in Equation (62) to get

$$\frac{\mathrm{d}}{\mathrm{d}H}\left[\frac{2\xi H}{\frac{\mathrm{d}\xi}{\mathrm{d}H}}\right] + \frac{\mathrm{d}\omega}{\mathrm{d}H} = 0. \tag{66}$$

Subsequent integration gives

$$\frac{\mathrm{d}\xi}{\mathrm{d}H} = \frac{2\xi H}{C_1 - \omega}, \tag{67}$$

where

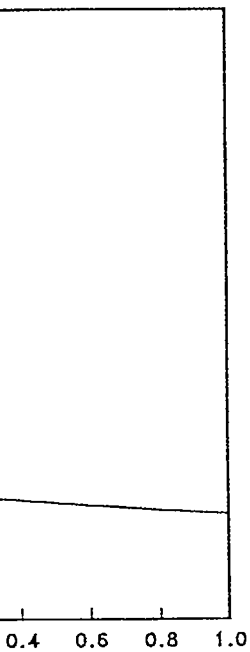
$$C_1 = \omega(0) \tag{68}$$

for the boundary condition

$$\left.\frac{\mathrm{d}\xi}{\mathrm{d}H}\right\|_{H=0} = \infty \tag{69}$$

to be satisfied. Integrating (67) once we obtain

$$\xi = C_2 \exp\left[2\int_0^H \frac{\beta}{\omega(0) - \omega(\beta)}\,\mathrm{d}\beta\right]. \tag{70}$$



of λ.

(66)

(67)

(68)

(69)

(70)

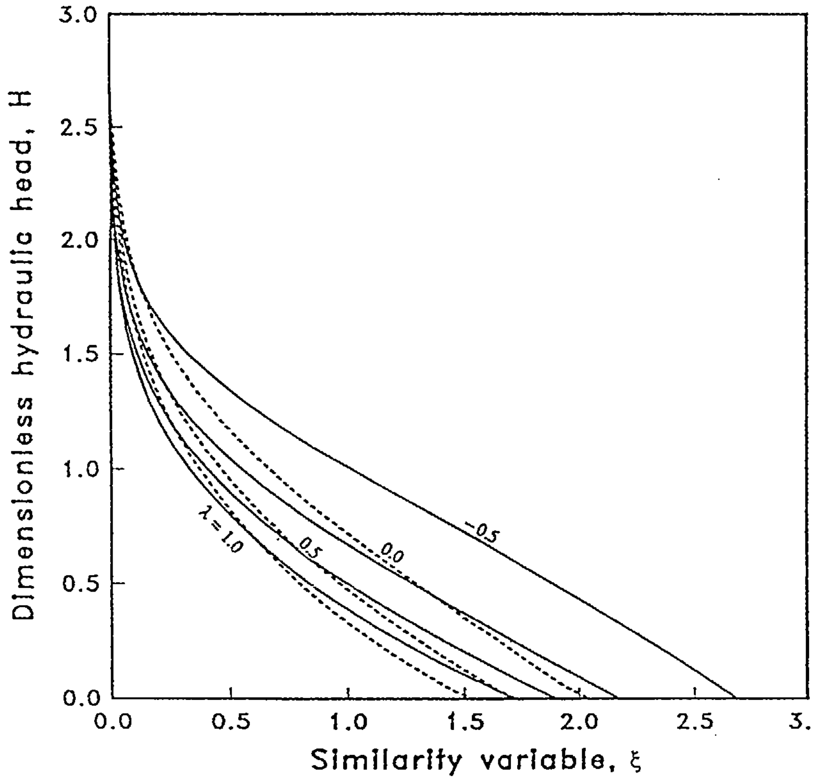


Fig. 10. The dimensionless hydraulic head profile function v for various values of λ. Dashed lines represent Barenblatt's solutions for the respective value of λ.

Substituting (67) and (70) into (65) results into

$$\frac{\mathrm{d}\omega}{\mathrm{d}H} = C_2^2 \left[\frac{1}{2} - \frac{2\lambda H^2}{\omega(0) - \omega(H)}\right] \exp\left[4 \int_0^H \frac{\beta}{\omega(0) - \omega(\beta)}\,\mathrm{d}\beta\right]. \tag{71}$$

A final integration yields the integral equation

$$\begin{aligned}\omega = {} & C_2^2 \int_0^H \left[\frac{1}{2} - \frac{2\lambda \alpha^2}{\omega(0) - \omega(\alpha)}\right] \\ & \exp\left[4 \int_0^\alpha \frac{\beta}{\omega(0) - \omega(\beta)}\,\mathrm{d}\beta\right]\,\mathrm{d}\alpha + C_3,\end{aligned} \tag{72}$$

where

$$C_3 = \omega(0). \tag{73}$$

Equation (72) is the radial flow analogue of (21). The other integration constant, C_2, can be determined from the boundary condition (63). We obtain

$$C_2 = \left[\int_0^\infty \left(\frac{1}{2} - \frac{2\lambda\alpha^2}{\omega(0)-\omega(\alpha)}\right)\right.$$
$$\left.\exp\left(4\int_0^\alpha \frac{\beta}{\omega(0)-\omega(\beta)}\,\mathrm{d}\beta\right)\,\mathrm{d}\alpha\right]^{-1/2}. \tag{74}$$

Substituting in (72) yields the final equation

$$\omega - \omega(0) = \frac{\int_0^H \left[\frac{1}{2} - \frac{2\lambda\alpha^2}{\omega(0)-\omega(\alpha)}\right]\exp\left[4\int_0^\alpha \frac{\beta}{\omega(0)-\omega(\beta)}\,\mathrm{d}\beta\right]\,\mathrm{d}\alpha}{\int_0^\infty \left[\frac{1}{2} - \frac{2\lambda\alpha^2}{\omega(0)-\omega(\alpha)}\right]\exp\left[4\int_0^\alpha \frac{\beta}{\omega(0)-\omega(\beta)}\,\mathrm{d}\beta\right]\,\mathrm{d}\alpha}. \tag{75}$$

As before, we can take $\omega(0)$ as zero or, equivalently, we may denote

$$f \equiv \omega - \omega(0) \tag{76}$$

to obtain the integral equation

$$f = \frac{\int_0^H \left[\frac{1}{2} + \frac{2\lambda\alpha^2}{f(\alpha)}\right]\exp\left[-4\int_0^\alpha \frac{\beta}{f(\beta)}\,\mathrm{d}\beta\right]\,\mathrm{d}\alpha}{\int_0^\infty \left[\frac{1}{2} + \frac{2\lambda\alpha^2}{f(\alpha)}\right]\exp\left[-4\int_0^\alpha \frac{\beta}{f(\beta)}\,\mathrm{d}\beta\right]\,\mathrm{d}\alpha}. \tag{77}$$

This equation is the analogue of (38). As in the rectilinear case, f can be identified with the relative total discharge function. From the solution of (77), the distribution of the dimensionless hydraulic head H can be determined. Substituting (74) and (76) in (70) gives

$$\xi = \frac{\exp\left[-2\int_0^H \frac{\beta}{f(\beta)}\,\mathrm{d}\beta\right]}{\left[\int_0^\infty \left[\frac{1}{2} + \frac{2\lambda\alpha^2}{f(\alpha)}\right]\exp\left[-4\int_0^\alpha \frac{\beta}{f(\beta)}\,\mathrm{d}\beta\right]\,\mathrm{d}\alpha\right]^{1/2}}. \tag{78}$$

In the special case $\lambda = 0$, Equations (77) and (78) reduce to those of Chen *et al.* (1991b)

$$f = \frac{\int_0^H \exp\left[-4\int_0^\alpha \frac{\beta}{f(\beta)}\,\mathrm{d}\beta\right]\,\mathrm{d}\alpha}{\int_0^\infty \exp\left[-4\int_0^\alpha \frac{\beta}{f(\beta)}\,\mathrm{d}\beta\right]\,\mathrm{d}\alpha} \tag{79}$$

and

$$\xi = \left[\frac{2\exp\left[-4\int_0^H \frac{\beta}{f(\beta)}\,\mathrm{d}\beta\right]}{\int_0^\infty \exp\left[-4\int_0^\alpha \frac{\beta}{f(\beta)}\,\mathrm{d}\beta\right]\,\mathrm{d}\alpha}\right]^{1/2}. \tag{80}$$

These authors studied the solution of the problem at constant injection rate.

To solve Equation (77) using an iterative procedure we take $f_0(H) = 1$ as the initial guess, as in Chen *et al.* (1991b). Numerical calculations were carried out for various values of λ in the range $-1 < \lambda \leqslant 1$. The relative total discharge function is plotted for several values of λ in Figures 7 and 8. Figure 9 shows the dimensionless front location, ξ_0, as a function of λ. Corresponding dimensionless hydraulic head profiles are shown in Figure 10, the dashed lines indicating the results by Barenblatt (1954). As can be seen, some discrepancy exists between the two solutions. This is in contrast to the rectilinear case, where the numerical results from both works are in excellent agreement. We suspect computational limitations, certainly greater in 1954 than today, as a possible source for this mismatch.

Conclusions

In this paper we have combined an integral equation formalism with a method of change of variables, generally known as the hodograph transformation, to solve a self-similar problem describing unsteady-state flow of groundwater in porous media. New, semi-analytical solutions were derived in both rectilinear and radial geometries for a family of problems, first formulated and investigated by Barenblatt. In the rectilinear case, our results and those of Barenblatt's coincide. In the radial case, we expect that our results are more precise. In both cases, the solutions developed here allow for additional analytical information and are considerably easier than previous works. We note that the present approach can be successfully applied to the solution of other nonlinear problems, including those in boundary layer theory, that admit a self-similar description.

Acknowledgements

This work was supported in part by the U.S. Department of Energy Contract No. DE-AC03-76SF0098, administered by the DOE Nevada Office, in cooperation with the United States Geological Survey, Denver.

References

Aronson, G., 1985, The porous media equation, some problems in nonlinear diffusion (A. Fasano and M. Primicerio, eds.), *Lecture Notes in Mathematics* (CIME Foundation Series), Springer-Verlag, No. 1224.

Barenblatt, G. I., 1952a, Some transient flow of fluid and gas in a porous medium (in Russian), *Prikladnaia Matematika i Mekhanika* **16**(1), 67–78.

Barenblatt, G. I., 1952b, Self-similar movements of compressible fluid in a porous medium (in Russian), *Prikladnaia Matematika i Mekhanika* **16**(6), 679–698.

Barenblatt, G. I., 1954, Some problems of transient flow of fluids through porous media (in Russian), *Izv. Akad. Nauk*, U.S.S.R., Otdel Tech. Nauk, No. 6, 97–110.

Barenblatt, G. I. and Vishek, M. I., 1956, On the finiteness of velocity of propagation in the problems of transient flow of liquid and gas in a porous medium (in Russian), *Prikladnaia Matematika i Mekhanika* **20**(3), 411–417.

Barenblatt, G. I., Entov, V. M. and Ryzhik, V. M., 1972, *Theory of Unsteady-State Flow of Liquid and Gas through Porous Media* (in Russian), 288pp., Nedra Moscow, pp. 69, 72.

Barenblatt, G. I., Entov, V. M. and Ryzhik, V. M., 1990, *Theory of Fluid Flows through Natural Rocks* 395pp., Kluwer, Dordrecht.

Bear, J., 1972, *Dynamics of Fluids in Porous Media*, American Elsevier, New York, NY, 763 pp.

Boussinesq, J., 1904, Recherches théoretiques sur l'écoulement des nappes d'eau infiltrées dans le sol et sur le debit des sources, *Jour. de Math. Pures et Appl.* **10**, 5–78.

Clarkson, P. A., Fokas, A. S. and Ablowitz, M. J., 1989, Hodograph transformations of linearizable partial differential equations, *SIAM J. Appl. Math.* **49**(4), 1188–1209.

Chen, Z.-X., Bodvarsson, G. S. and Witherspoon, P. A., 1990, An integral equation formulation for two-phase flow and other nonlinear flow problems through porous media, paper SPE 20517 presented at the 1990 SPE Annual Fall Meeting, New Orleans, LA, Sept. 23–26.

Chen, Z.-X., Bodvarsson, G. S. and Witherspoon, P. A., 1991a, One-dimensional horizontal infiltration in an unsaturated porous media including air viscosity and applied pressure, preprint.

Chen, Z.-X., Bodvarsson, G. S. and Witherspoon, P. A., 1991b, Exact semi-analytical solution for unconfined Dupuit flow of groundwater, preprint.

van Duijn, C. J. and Peletier, L. A., 1992, A boundary-layer problem in fresh-salt groundwater flow, *Q.Jl. Mech. appl. Math.* **45**, 1–23.

Fokas, A. S. and Yortsos, Y. C., 1982, On the exactly solvable equation $S_t = [(\beta S + \lambda)^{-2} S_x]_x + \alpha(\beta S + \lambda)^{-2} S_x$ occurring in two-phase flow in porous media, *SIAM J. Appl. Math.* **42**(2) 318.

Lake, L. W., 1989, *Enhanced Oil Recovery*, Prentice Hall, New York.

Leibenson, L. S., 1929, Gas-flow in a porous medium (in Russian), *Neftianoe Khoziaistvo* **10**, 497–519.

Lenormand, R., 1990, Liquids in Porous Media, *J. Phys.: Condens. Matter* **3**, SA79–89.

McWhorter, D. B., 1990, Unsteady radial flow of gas in the vadose zone, *Journal of Contaminant Hydrology* **5**(3), 297–314.

McWhorter, D. B. and Sunada, D. K., 1990, Exact integral solutions for two-phase flow, *Water Resour. Res.* **26**(3), 399–413.

Polubarinova-Kochina, P. Ya, 1948, A nonlinear partial differential equation encountered in theory of fluid flow through porous media (in Russian), *Dokl. Akad. Nauk, U.S.S.R.* **63**(6), 623–626.

Polubarinova-Kochina, P. Ya, 1949, Unsteady movements of groundwater when it flows from a reservoir (in Russian), *Prikladnaia Matematika i Mekhanika* **13**(2), 187–206.

Shankar, K. and Yortsos, Y. C., 1983, Asymptotic analysis of single pore gas-solid reactions, *Chem. Engng. Sci.* **38**(8), 1159–1165.

Yortsos, Y. C., 1991, A theoretical analysis of Vertical Flow Equilibrium, paper SPE 22612 presented at the 66th SPE Annual Fall Meeting, Dallas, TX, Oct. 6–9. Also, *Transport in Porous Media*, in press (1995).